

COMPACT MANIFOLDS WITH FIXED BOUNDARY AND LARGE STEKLOV EIGENVALUES

BRUNO COLBOIS, AHMAD EL SOUFI, AND ALEXANDRE GIROUARD

ABSTRACT. Let (M, g) be a compact Riemannian manifold with boundary. Let $b > 0$ be the number of connected components of its boundary. For manifolds of dimension ≥ 3 , we prove that for $j = b + 1$ it is possible to obtain an arbitrarily large Steklov eigenvalue $\sigma_j(M, e^\delta g)$ using a conformal perturbation $\delta \in C^\infty(M)$ which is supported in a thin neighbourhood of the boundary, with $\delta = 0$ on the boundary. For $j \leq b$, it is also possible to obtain arbitrarily large eigenvalues, but the conformal factor must spread throughout the interior of M . In fact, when working in a fixed conformal class and for $\delta = 0$ on the boundary, it is known that the volume of $(M, e^\delta g)$ has to tend to infinity in order for some σ_j to become arbitrarily large. This is in stark contrast with the situation for the eigenvalues of the Laplace operator on a closed manifold, where a conformal factor that is large enough for the volume to become unbounded results in the spectrum collapsing to 0. We also prove that it is possible to obtain large Steklov eigenvalues while keeping different boundary components arbitrarily close to each other, by constructing a convenient Riemannian submersion.

1. INTRODUCTION

The Steklov eigenvalues of a smooth compact connected Riemannian manifold (M, g) of dimension $n + 1 \geq 2$ with boundary Σ are the real numbers σ for which there exists a non-zero harmonic function $f : M \rightarrow \mathbb{R}$ which satisfies $\partial_\nu f = \sigma f$ on the boundary Σ . Here and further ∂_ν is the outward normal derivative on Σ . It is well known that the Steklov eigenvalues form a discrete spectrum $0 = \sigma_1 < \sigma_2 \leq \sigma_3 \leq \dots \nearrow \infty$, where each eigenvalue is repeated according to its multiplicity. The interplay between the geometry of M and the Steklov spectrum has recently attracted substantial attention. See [8] and the references therein for recent development and open problems.

During the first week of 2017, AG and BC were supposed to travel to Tours and work with Ahmad El Soufi to complete this paper. We learned just a few days before our visit of his untimely death. Ahmad was a colleague and a friend. He will be dearly missed.

Many developments linking the Steklov eigenvalues of a compact manifold M with the eigenvalues λ_j of the Laplace operator on its boundary have appeared. See for instance [13, 3] and more recently [12], where it was proved that for any Euclidean domain $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary, there exists a constant $c_\Omega > 0$ such that

$$\lambda_j \leq \sigma_j^2 + 2c_\Omega \sigma_j, \quad \text{and} \quad \sigma_j \leq c_\Omega + \sqrt{c_\Omega^2 + \lambda_j}.$$

These results indicate a strong link between the Steklov eigenvalues of a manifold and the geometry of its boundary. In fact, on smooth surfaces the spectral asymptotics is completely determined by the geometry of the boundary [7].

On the other hand, Steklov eigenvalues are not very sensitive to perturbation away from the boundary. For instance given any domain $\Omega \subset M$ with compact closure (at a positive distance from the boundary), it is easy to see that there exists a constant C_Ω such that any Riemannian metric g' which coincides with g on $M \setminus \Omega$ satisfies

$$|\sigma_j(M, g') - \sigma_j(M, g)| \leq C_\Omega \quad \text{for each } j \in \mathbb{N}.$$

See for instance Proposition 1.2.

In the present paper, we investigate the following question:

For a given closed Riemannian manifold Σ , how large can $\sigma_2(M)$ be among compact Riemannian manifolds M with boundary isometric to Σ ?

For a manifold (M, g) of dimension $n + 1 \geq 3$ with boundary Σ , we will see that it is possible to make σ_2 arbitrarily large by using conformal perturbations $g' = h^2g$ such that $h = 1$ on Σ . Of course, this imply that any eigenvalue σ_j becomes arbitrarily large under such a conformal perturbation, but the situation is more interesting than that. Indeed, let $b > 0$ be the number of connected components. We will prove that it is possible to make σ_{b+1} arbitrarily large by using a conformal perturbation h^2g where h is a smooth function which is different from 1 only in a thin strip located arbitrarily close to the boundary (with $h = 1$ identically on Σ). It is also possible to make lower eigenvalues σ_j arbitrarily large, but this requires conformal perturbations which penetrates deeply into the manifold M .

One could also ask how small an eigenvalue $\sigma_j(M)$ can be. This question is easier, as it is relatively easy to construct small eigenvalues while keeping the boundary fixed. On surfaces, it is sufficient to create thin passages (see Figure 3 of [8, Section 4]) while for manifolds of dimension ≥ 3 , one can use a conformal perturbation supported inside the manifold M . See Proposition 2.1.

Large eigenvalues on surfaces. It was proved in [11] that any compact surface M with boundary of length $L > 0$ satisfies

$$\sigma_2(M) \leq \frac{8\pi}{L}(1 + \text{genus}(M)).$$

In [4], a sequence of surfaces $(M_l)_{l \in \mathbb{N}}$ with one boundary component of fixed length $L > 0$ was constructed, which satisfies

$$\lim_{l \rightarrow \infty} \sigma_2(M_l) = +\infty.$$

These two results give a complete answer to our initial question for surfaces: it is possible to obtain arbitrarily large σ_2 , but it is necessary to increase the genus of M in order to do so.

Manifolds of higher dimensions. For any compact Riemannian manifold (M, g) of dimension ≥ 3 with boundary Σ , we will show that a conformal perturbation is sufficient to obtain arbitrarily large σ_2 . More can be said: let $b > 0$ be the number of connected components of the boundary Σ . The next theorem shows that it is possible to make σ_{b+1} arbitrarily large using conformal perturbations g_ϵ which are supported in an arbitrary neighbourhood of the boundary Σ , and which coincide with g on the boundary. It is also possible to make σ_2 large, but this requires conformal perturbations away from the boundary (See Proposition 3.2). The next theorem is the main result of this paper.

Theorem 1.1. *Let (M, g) be a compact connected Riemannian manifold of dimension ≥ 3 with $b \in \mathbb{N}$ boundary components $\Sigma_1, \dots, \Sigma_b$.*

(i) For every neighborhood V of Σ , there exists a one-parameter family of Riemannian metrics g_ϵ conformal to g which coincide with g on Σ and in $M \setminus V$, such that

$$\sigma_{b+1}(g_\epsilon) \rightarrow \infty \quad \text{as} \quad \epsilon \rightarrow 0.$$

(ii) There exists a one-parameter family of Riemannian metrics g_ϵ conformal to g which coincide with g on Σ such that

$$\sigma_2(g_\epsilon) \rightarrow \infty \quad \text{as} \quad \epsilon \rightarrow 0.$$

The proof of Theorem 1.1 will be presented in Section 3. It is important to note that in order to obtain large eigenvalues, it is necessary to perturb the metric near each points of the boundary.

Proposition 1.2. *Let (M, g) be a compact Riemannian manifold with boundary Σ . Let $p \in \Sigma$ and let $\epsilon > 0$. Then any Riemannian metric g' on M which coincide with g on $B(p, \epsilon) \subset M$ satisfy*

$$\lambda_k^N(B(p, \epsilon), g) \leq \sigma_k(M, g') \leq \lambda_k^D(B(p, \epsilon), g),$$

where $\lambda_k^D(B(p, \epsilon), g)$ and $\lambda_k^N(B(p, \epsilon), g)$ are the k -th eigenvalue of a mixed Steklov-Dirichlet eigenvalue and Steklov-Neumann eigenvalue problems respectively.

The proof of this observation is an exercise in the use of min-max characterisations of eigenvalues. It will be presented in Section 2.

The conformal perturbations which are used in the proof of Theorem 1.1 are such that the volume $|M|_{g_\epsilon}$ tends to infinity as $\epsilon \rightarrow 0$. This is a necessary condition when working in a fixed conformal class $[g]$. Indeed, the following inequality for $g' \in [g]$ was proved in [9]:

$$\sigma_k(M, g') |\Sigma|^{1/n} < \frac{A + Bk^{\frac{2}{n+1}}}{I(M)^{(n-1)/n}},$$

where A is a constant which depends on g , B depends on the dimension and $I(M)$ is the isoperimetric ratio

$$I(M) = \frac{|\Sigma|_{g'}}{|M|_{g'}^{n/(n+1)}}.$$

In each of the constructions, the diameter also becomes unbounded. In Theorem 3.5, we construct a sequence g_m of Riemannian metrics on M such that $g_m|_\Sigma = g_\Sigma$ and such that (M, g_m) has uniformly bounded diameter and σ_2 becomes arbitrarily large.

To conclude this introduction, note that it is difficult to obtain lower bounds for Steklov eigenvalues. Under relatively strong convexity assumptions this was already investigated by Escobar in [6]. More recently Jammes [10] proposed an interesting inequality in the spirit of Cheeger:

$$\sigma_2(M) \geq \frac{h(M)j(M)}{4}.$$

Here $h(M)$ is the classical Cheeger constant and $j(M)$ was introduced in [10]. It is challenging to obtain effective lower bounds on $\sigma_2(M)$ using this inequality, mainly because it is difficult to estimate $h(M)$ and $j(M)$. Moreover, it is interesting to note that the metrics which we construct in Theorem 1.1 and 3.5 have small Cheeger and Cheeger-Jammes constants, despite the eigenvalue σ_2 being arbitrarily large.

1.1. Plan of the paper. In the next section, we present the variational characterization of the Steklov and mixed Steklov-Dirichlet eigenvalue problems and deduce some simple consequences. In Section 3 we prove the main results of the paper by first working in cylinders

and then using quasi-isometric control of eigenvalues to obtain Theorem 1.1. We also prove Theorem 3.5 which provides an example where two boundary components are arbitrarily close to each other.

2. VARIATIONAL CHARACTERISATION AND QUASI-ISOMETRIC CONTROL OF EIGENVALUES

Let M be a smooth compact Riemannian manifold with boundary Σ . Let $\mathcal{H}_k(M)$ be the set of k -dimensional linear subspaces of $C^\infty(M)$. It is well known that the Steklov eigenvalue σ_k is given by

$$\sigma_k(M, g) = \min_{E \in \mathcal{H}_k} \max_{0 \neq f \in E} \frac{\int_M |df|_g^2 dv_g}{\int_\Sigma |f|^2 dv_g}, \quad (1)$$

where dv_g is the volume form. In the following we will use conformal metrics of the form $g' = h^2 g$, where h is a smooth function on M such that $h = 1$ identically on the boundary Σ . In the min-max characterization of $\sigma_k(M, h^2 g)$, the denominator is the same as above, while the numerator is

$$\int_M |df|_{g'}^2 dv_{g'} = \int_M |df|_g^2 h^{n-1} dv_g.$$

In the following, we will often write $|df|$ for $|df|_g$.

We seize this opportunity to prove one of the simple statement from the introduction.

Proposition 2.1. *Let M be a compact smooth Riemannian manifold of dimension $n + 1$ at least 3, with boundary Σ . For each $p \in \Sigma$ and each $\epsilon > 0$, there exists a sequence of conformal deformations $h_m^2 g$ such that $h_m > 0$ is a smooth function which is identically equal to 1 on Σ and on the complement of the ball $B(p, \epsilon) \subset M$, and such that $\lim_{m \rightarrow \infty} \sigma_k(M, h_m^2 g) = 0$ for each $k \in \mathbb{N}$.*

Proof. Given $\epsilon > 0$, let $p \in \Sigma$ and consider a smooth function $f \in C^\infty(M)$ which is supported in $B(p, \epsilon) \subset M$ and which does not vanish at p . Let h_m be a sequence of positive smooth functions on M such that $h_m = 1$ on Σ and $\lim_{m \rightarrow \infty} h_m = 0$ uniformly on compact subsets of $B(p, \epsilon) \cap \text{interior}(M)$. It follows that the conformal deformations $\tilde{g}_m = h_m^2 g$ satisfy

$$\lim_{m \rightarrow \infty} \frac{\int_M |df|_{\tilde{g}_m}^2 dv_{\tilde{g}_m}}{\int_\Sigma f^2 dv_{\tilde{g}_m}} = \frac{\int_M |df|_g^2 h_m^{n-1} dv_g}{\int_\Sigma f^2 dv_g} = 0.$$

Using k functions f_j ($j = 1, \dots, k$) with disjoint support in $B(p, \epsilon)$ instead of a single function f , the result now follows from the min-max characterization of σ_k . \square

We will also use the following mixed Steklov-Dirichlet problem on a domain $\Omega \subset M$:

$$\Delta f = 0 \text{ in } \Omega, \quad f = 0 \text{ on } \partial\Omega \setminus \Sigma, \quad \partial_\nu f = \lambda f \text{ on } \partial\Omega \cap \Sigma.$$

It has discrete spectrum $0 < \lambda_1^D \leq \lambda_2^D \leq \dots \nearrow \infty$. The k -th eigenvalue is given by

$$\lambda_k^D = \min_{E \in \mathcal{H}_{k,0}} \max_{0 \neq f \in E} \frac{\int_\Omega |\nabla f|^2 dv_g}{\int_{\partial\Omega \cap \Sigma} f^2 dv_g}, \quad (2)$$

where $\mathcal{H}_{k,0} = \{E \in \mathcal{H}_k : f = 0 \text{ on } \partial\Omega \setminus \Sigma \times \{0\} \ \forall f \in E\}$. For more informations on mixed Steklov problems see for instance [2] and [1].

We are now ready for the proof of Proposition 1.2.

Proof of Proposition 1.2. Let (ϕ_k) be a sequence of eigenfunctions corresponding to $\lambda_k^D(B(p, \epsilon))$, which are extended by 0 elsewhere in M . Using the subspace $E_k = \text{span}(\phi_1, \dots, \phi_k)$ in the min-max characterization of $\sigma_k(M, g')$ completes the proof. \square

The following proposition is borrowed from [4]. It is classical and follows directly from the min-max characterization of the eigenvalues. We believe that this principle was used for the first time in [5], in the context of the Laplace-Beltrami operator acting on differential forms.

Proposition 2.2. *Let M be a compact manifold of dimension n , with smooth boundary Σ and let g_1, g_2 be two Riemannian metrics on M which are quasi-isometric with ratio $A \geq 1$, which means that for each $x \in M$ and $0 \neq v \in T_x M$ we have*

$$\frac{1}{A} \leq \frac{g_1(x)(v, v)}{g_2(x)(v, v)} \leq A.$$

Then the Steklov eigenvalues with respect to g_1 and g_2 satisfies the following inequality:

$$\frac{1}{A^{2n+1}} \leq \frac{\sigma_k(M, g_1)}{\sigma_k(M, g_2)} \leq A^{2n+1}.$$

Note also that if the metrics g_1 and g_2 are quasi-isometric with ratio $A \geq 1$, then given a smooth function h on M , the two conformal metrics $h^2 g_1$ and $h^2 g_2$ are also quasi-isometric with the same ratio A . This will be useful in the proof of Theorem 1.1 when going from cylindrical boundaries to arbitrary manifolds.

3. LARGE STEKLOV EIGENVALUES ON MANIFOLDS WITH FIXED BOUNDARY

Let M be a compact manifold of dimension $(n + 1)$ with $b \geq 1$ boundary components: $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_b$. We will prove Theorem 1.1 by first working under the extra hypothesis that the boundary Σ of M has a neighbourhood which is isometric to the product $\Sigma \times [0, L)$ for some $L > 0$. This is not a strong hypothesis since it is always satisfied up to a quasi-isometry (See the proof of Theorem 1.1 below).

In the present context, we note g_0 the restriction of the Riemannian metric g to the boundary Σ , and d_0 the corresponding exterior derivative on $C^\infty(\Sigma)$. The spectrum of the Laplace operator on Σ is denoted

$$0 = \lambda_1 = \dots = \lambda_b < \lambda_{b+1} \leq \dots \rightarrow +\infty.$$

Theorem 1.1 will follow from Proposition 3.1 and Proposition 3.3 below.

Proposition 3.1. *Let (M, g) be a Riemannian manifold of dimension $n+1 \geq 3$, with boundary Σ and assume that there exists a neighborhood V of Σ which is isometric to the product $\Sigma \times [0, L)$ for some $L > 0$. For every $\varepsilon > 0$ sufficiently small, there exists a Riemannian metric $g_\varepsilon = h_\varepsilon^2 g$ conformal to g which coincides with g in the complement of $\Sigma \times (\varepsilon, 4\varepsilon)$ and such that*

$$\sigma_{b+1}(g_\varepsilon) \geq \frac{A}{\varepsilon},$$

where $A = \frac{1}{4} \min\{\lambda_{b+1}(\Sigma), \frac{1}{4}\} > 0$.

Proof. For every positive $\varepsilon < \min\{\frac{L}{4}, \frac{2}{L}\}$, define a Riemannian metric $g_\varepsilon = h_\varepsilon^2 g$ on M where $h_\varepsilon \geq 1$ is a smooth function which is identically equal to 1 in the complement of $\Sigma \times (\varepsilon, 4\varepsilon)$ and, for $(x, t) \in \Sigma \times [2\varepsilon, 3\varepsilon]$,

$$h_\varepsilon(x, t) = \varepsilon^{-2}.$$

Let $\{\phi_k\}_{k \in \mathbb{N}}$ be an orthonormal basis of eigenfunctions of the Laplacian on Σ , with $\Delta \phi_k = \lambda_k \phi_k$. Denote by $\Sigma_1, \dots, \Sigma_b$ the connected components of Σ . One has $\lambda_1 = \dots = \lambda_b = 0$ and, for every $j \leq b$, one chooses $\phi_j = |\Sigma_j|^{-\frac{1}{2}}$ on Σ_j and $\phi_j = 0$ elsewhere.

Let f be a smooth function on M with $\int_\Sigma f dv_{g_0} = 0$ and $\int_\Sigma f^2 dv_{g_0} = 1$. The restriction of f to $\Sigma \times [0, L)$ is developed in Fourier series:

$$f(x, t) = \sum_{j \geq 1} a_j(t) \phi_j(x)$$

with

$$a_j(0) = |\Sigma_j|^{-\frac{1}{2}} \int_{\Sigma_j} f dv_{g_0}, \quad \text{for } j = 1, \dots, b$$

and, since $\int_{\Sigma} f dv_{g_0} = 0$ and $\int_{\Sigma} f^2 dv_{g_0} = 1$

$$a_1(0)|\Sigma_1|^{\frac{1}{2}} + \cdots a_b(0)|\Sigma_b|^{\frac{1}{2}} = 0 \quad (3)$$

$$\sum_{j \geq 1} a_j(0)^2 = 1. \quad (4)$$

Observe that $\sum_{j=1}^b a_j(0)^2$ is the square of the L^2 -norm of the orthogonal projection of $f|_{\Sigma}$ on $\ker(\Delta) = \text{span}\{\phi_1, \dots, \phi_b\}$, in $L^2(\Sigma, g_0)$. From

$$df(x, t) = \sum_{j \geq 1} (a'_j(t)\phi_j(x) dt + a_j(t) d_0\phi_j(x))$$

and $\int_{\Sigma} |d_0\phi_j|^2 dv_{g_0} = \lambda_j$, it follows that the Dirichlet energy of f on $(M, h_{\epsilon}^2 g)$ is

$$\begin{aligned} R_{\epsilon}(f) &:= \int_M |df|^2 h_{\epsilon}^{n-1} dv_g \geq \int_{\Sigma \times (0, L)} |df|^2 h_{\epsilon}^{n-1} dv_g \\ &\geq \sum_{j \geq 1} \int_0^L (a'_j(t)^2 + \lambda_j a_j(t)^2) h_{\epsilon}^{n-1}(t) dt. \end{aligned} \quad (5)$$

At this point, observe that either the function a_j decreases quickly when moving away from the boundary (which costs energy from the first term in (5)) or it remains big enough, and the second term contributes a large amount to the energy $R_{\epsilon}(f)$. This is now explained more precisely.

Fix an integer $j \geq 1$. If $|a_j(t)| \geq \frac{1}{2}|a_j(0)|$ for all $t \in [2\epsilon, 3\epsilon]$, then

$$\begin{aligned} \int_0^L \lambda_j a_j(t)^2 h_{\epsilon}^{n-1}(t) dt &\geq \lambda_j \int_{2\epsilon}^{3\epsilon} a_j(t)^2 \epsilon^{-2(n-1)} dt \\ &\geq \frac{\lambda_j}{4} \epsilon^{-2n+3} a_j(0)^2 \geq \frac{\lambda_j}{4\epsilon} a_j(0)^2. \end{aligned} \quad (6)$$

Otherwise, there exists $t_0 \in [2\epsilon, 3\epsilon]$ with $|a_j(t_0)| \leq \frac{1}{2}|a_j(0)|$, which implies $|a_j(0) - a_j(t_0)| \geq |a_j(0)| - |a_j(t_0)| \geq \frac{1}{2}|a_j(0)|$ and using the Cauchy-Schwarz inequality this leads to

$$\begin{aligned} \int_0^L a'_j(t)^2 h_{\epsilon}^{n-1}(t) dt &\geq \int_0^{2\epsilon} a'_j(t)^2 dt + \int_{2\epsilon}^{t_0} a'_j(t)^2 \epsilon^{-2(n-1)} dt \\ &\geq \frac{1}{2\epsilon} \left(\int_0^{2\epsilon} a'_j(t) dt \right)^2 + \frac{\epsilon^{-2(n-1)}}{t_0 - 2\epsilon} \left(\int_{2\epsilon}^{t_0} a'_j(t) dt \right)^2 \end{aligned} \quad (7)$$

where $\frac{\varepsilon^{-2(n-1)}}{t_0-2\varepsilon} \geq \frac{\varepsilon^{-2(n-1)}}{\varepsilon} = \frac{1}{\varepsilon} \geq \frac{1}{2\varepsilon}$. Hence (since $x^2 + y^2 \geq \frac{1}{2}(x+y)^2$)

$$\begin{aligned} \int_0^L a'_j(t)^2 h_\varepsilon^{n-1}(t) dt &\geq \frac{1}{2\varepsilon} \left[\left(\int_0^{2\varepsilon} a'_j(t) dt \right)^2 + \left(\int_{2\varepsilon}^{t_0} a'_j(t) dt \right)^2 \right] \\ &\geq \frac{1}{4\varepsilon} \left(\int_0^{2\varepsilon} a'_j(t) dt + \int_{2\varepsilon}^{t_0} a'_j(t) dt \right)^2 \\ &= \frac{1}{4\varepsilon} (a_j(0) - a_j(t_0))^2 \geq \frac{1}{16\varepsilon} a_j(0)^2. \end{aligned} \quad (8)$$

For $j \geq b+1$ one has $\lambda_j \geq \lambda_{b+1}$ and, combining (6) and (8) leads to

$$\int_0^L (a'_j(t)^2 + \lambda_j a_j(t)^2) h_\varepsilon^{n-1}(t) dt \geq \min\left\{\frac{\lambda_{b+1}}{4}, \frac{1}{16}\right\} \frac{1}{\varepsilon} a_j(0)^2 = \frac{A}{\varepsilon} a_j(0)^2.$$

Therefore, thanks to (5) and (4),

$$R_\varepsilon(f) \geq \frac{A}{\varepsilon} \sum_{j \geq b+1} a_j(0)^2 = \frac{A}{\varepsilon} \left(1 - \sum_{j \leq b} a_j(0)^2 \right). \quad (9)$$

Now, every normalized function f which is orthogonal in $L^2(\Sigma)$ to $\ker(\Delta) = \text{span}\{\phi_1, \dots, \phi_b\}$, satisfies $\sum_{j \leq b} a_j(0)^2 = 0$ and, then $R_\varepsilon(f) \geq \frac{A}{\varepsilon}$. Using the max-min principle we deduce that $\sigma_{b+1}(M, g_\varepsilon) \geq \frac{A}{\varepsilon}$. This completes the proof of (i). \square

In Proposition 3.3 below, we will prove that it is also possible to make σ_2 arbitrarily large using a conformal perturbation $h^2 g$. This is more difficult than for σ_{b+1} . One of the difficulties comes from the fact that the conformal perturbation will need to be supported everywhere inside the manifold M . This follows from the following easy proposition.

Proposition 3.2. *For every Riemannian metric g' which coincides with g on Σ and in the complement of $\Sigma \times [0, \frac{L}{2})$, one has*

$$\sigma_b(g') \leq \frac{2}{L}.$$

Proof of Proposition 3.2. For every $j \leq b$, let ψ_j be the function on M such that ψ_j is constant equal to zero in the complement of $\Sigma_j \times [0, L)$ and, for $(x, t) \in \Sigma_j \times [0, L)$,

$$\psi_j(x, t) = \begin{cases} |\Sigma_j|^{-\frac{1}{2}} & \text{in } \Sigma_j \times [0, \frac{L}{2}], \\ 2(1 - \frac{t}{L})|\Sigma_j|^{-\frac{1}{2}} & \text{in } \Sigma_j \times [\frac{L}{2}, L]. \end{cases}$$

For any Riemannian metric g' which coincides with g on Σ and on the complement of $\Sigma \times [0, \frac{L}{2}]$, one has $\int_M |d\psi_j|_{g'}^2 dv_{g'} = \frac{2}{L}$ and $\int_\Sigma \psi_j^2 dv_{g'} =$

1. Moreover, ψ_1, \dots, ψ_b are mutually orthogonal on the boundary. Therefore, using the min-max principle we deduce that $\sigma_b(g') \leq \frac{2}{L}$. \square

Proposition 3.3. *Let (M, g) be a Riemannian manifold of dimension $n+1 \geq 3$ with boundary Σ and assume that there exists a neighbourhood of Σ which is isometric to the product $\Sigma \times [0, L)$ for some $L > 0$. For every $\varepsilon > 0$ sufficiently small, there exists a Riemannian metric $g_\varepsilon = h_\varepsilon^2 g$ conformal to g which coincides with g in the neighbourhood $\Sigma \times [0, \varepsilon)$ of Σ and such that*

$$\sigma_2(g_\varepsilon) \geq \frac{C}{\varepsilon}$$

where C is an explicit constant which only depends on g .

The following Poincaré type result will be useful.

Lemma 3.4. *Let (M, g) be a compact manifold and denote by μ the first positive eigenvalue of the Laplacian of (M, g) with Neumann boundary condition if ∂M is nonempty. Let V_1 and V_2 be two disjoint measurable subsets of M of positive volume. Every function $f \in C^\infty(M)$ satisfies*

$$\int_M |df|^2 dv_g \geq \frac{\mu}{2} \min(|V_1|_g, |V_2|_g) \left(\int_{V_1} f dv_g - \int_{V_2} f dv_g \right)^2.$$

where $\int_M f dv_g := \frac{1}{|M|_g} \int_M f dv_g$.

Proof of Lemma 3.4. Denote by $m = \int_M f dv_g$ the mean value of f on M . The function $f - m$ is orthogonal to constant functions on M which implies

$$\int_M |df|^2 dv_g \geq \mu \int_M (f - m)^2 dv_g \geq \mu \int_{V_1} (f - m)^2 dv_g + \mu \int_{V_2} (f - m)^2 dv_g.$$

Using the Cauchy-Schwarz inequality, we get for $j = 1, 2$,

$$\int_{V_j} (f - m)^2 dv_g \geq \frac{1}{|V_j|} \left(\int_{V_j} (f - m) dv_g \right)^2 = |V_j| \left(\int_{V_j} f dv_g - m \right)^2$$

and then (since $x^2 + y^2 \geq \frac{1}{2}(x - y)^2$)

$$\int_{V_1} (f - m)^2 dv_g + \int_{V_2} (f - m)^2 dv_g \geq \frac{1}{2} \min(|V_1|, |V_2|) \left(\int_{V_1} f dv_g - \int_{V_2} f dv_g \right)^2$$

which ends the proof. \square

The proof of Proposition 3.3 is more subtle than that of Proposition 3.1. The behaviour of a smooth function f away from the boundary has to be taken into account in this case. Indeed, the Steklov eigenfunctions corresponding to index $j \leq b$ can be almost constant on connected components of the boundary. They do not spend a lot of energy "laterally". This is expressed by the situation where $\sum_{j=1}^b a_j(0)^2$ is large in the proof below.

Proof of Proposition 3.3. The situation where the boundary is connected is already treated by Proposition 3.1. Therefore we assume that $b \geq 2$ in what follows. As in the proof of Proposition 3.1, we define for every positive $\varepsilon < \min\{\frac{L}{4}, \frac{2}{L}\}$, a Riemannian metric $g_\varepsilon = h_\varepsilon^2 g$ on M where $h_\varepsilon \geq 1$ is a smooth function which is identically equal to ε^{-2} in $M \setminus (\Sigma \times [0, L])$ and identically equal to 1 on $\Sigma \times [0, \varepsilon]$. Note that unlike the conformal deformation used in the proof of Proposition 3.1, here, the metric g_ε tends to infinity everywhere in the interior of M as $\varepsilon \rightarrow 0$. Let f be a smooth function on M with $\int_\Sigma f dv_{g_0} = 0$ and such that $\int_\Sigma f^2 dv_{g_0} = 1$. Our goal is to get a lower bound for the Dirichlet energy $R_\varepsilon(f) = \int_M |df|_{g_\varepsilon}^2 dv_{g_\varepsilon}$ of the form C/ε .

Let $\Sigma_1, \dots, \Sigma_b$ be the connected components of the boundary Σ . As in the proof of Proposition 3.1, we consider an orthonormal basis $\{\phi_j\}_{j \in \mathbb{N}}$ of eigenfunctions of the Laplacian on Σ , with $\Delta \phi_j = \lambda_j \phi_j$, $\lambda_1 = \dots = \lambda_b = 0$ and, for every $j \leq b$, $\phi_j = |\Sigma_j|^{-\frac{1}{2}}$ on Σ_j and $\phi_j = 0$ elsewhere. The restriction of f to $\Sigma \times [0, L]$ is

$$f(x, t) = \sum_{j \geq 1} a_j(t) \phi_j(x).$$

In what follows we treat separately the case where $\sum_{j=1}^b a_j(0)^2$ is small and the case where it is large. Indeed, following step by step the proof of Proposition 3.1, we get

$$R_\varepsilon(f) \geq \frac{A}{\varepsilon} \left(1 - \sum_{j \leq b} a_j(0)^2 \right)$$

with $A = \frac{1}{4} \min\{\lambda_{b+1}, \frac{1}{4}\} > 0$. Hence, if $\sum_{j \leq b} a_j(0)^2 \leq \frac{1}{2}$, then

$$R_\varepsilon(f) \geq \frac{A}{2\varepsilon}. \tag{10}$$

Assume now that $\sum_{j \leq b} a_j(0)^2 \geq \frac{1}{2}$ and recall that the number of boundary components $b \geq 2$. We will prove that two of the boundary components, w.l.o.g. Σ_1 and Σ_2 , are such that $a_1(0) > 0$ is large enough while $a_2(0) < 0$ is small. Proceeding as in the proof of Proposition 3.1,

we will first treat the case where both $|a_1(t)|$ and $|a_2(t)|$ decreases quickly away from the boundary. Otherwise, we will appeal to Lemma 3.4. More precisely, let Ω be the complement in M of $\Sigma \times [0, \frac{L}{2})$. We will prove that

$$R_\varepsilon(f) \geq \frac{B}{2\varepsilon} \quad (11)$$

with

$$B = \frac{\min\{\mu(\Omega, g)bL, \frac{1}{2b}\}}{32(b-1)^2} \frac{\min_{j \leq b} |\Sigma_j|}{\max_{j \leq b} |\Sigma_j|},$$

where, $\mu(\Omega, g)$ is the first positive Neumann eigenvalue of the Laplacian in Ω .

Indeed, from $\sum_{j \leq b} a_j(0)^2 \geq \frac{1}{2}$ we deduce the existence of $j_0 \leq b$ with $a_{j_0}(0)^2 \geq \frac{1}{2b}$. We assume w.l.o.g. that $j_0 = 1$ and that $a_1(0) \geq \frac{1}{\sqrt{2b}}$. From (3) one has $\sum_{2 \leq j \leq b} a_j(0)|\Sigma_j|^{\frac{1}{2}} = -a_1(0)|\Sigma_1|^{\frac{1}{2}} \leq -\frac{|\Sigma_1|^{\frac{1}{2}}}{\sqrt{2b}}$. This implies w.l.o.g. that $a_2(0)|\Sigma_2|^{\frac{1}{2}} \leq -\frac{1}{b-1}a_1(0)|\Sigma_1|^{\frac{1}{2}} \leq -\frac{|\Sigma_1|^{\frac{1}{2}}}{(b-1)\sqrt{2b}}$.

Now, as in (7) and (8), if there exists $t_0 \in [\frac{L}{2}, L) \subset [2\varepsilon, L)$ with $a_1(t_0) \leq \frac{1}{2}a_1(0)$, we would have (with $a_1(0) - a_1(t_0) \geq \frac{1}{2}a_1(0)$)

$$\begin{aligned} R_\varepsilon(f) &\geq \int_0^{t_0} a_1'(t)^2 h_\varepsilon^{n-1}(t) dt \\ &= \int_0^{2\varepsilon} a_1'(t)^2 dt + \int_{2\varepsilon}^{t_0} a_1'(t)^2 \varepsilon^{-2(n-1)} dt \\ &\geq \frac{1}{16\varepsilon} a_1(0)^2 \geq \frac{1}{32b\varepsilon}. \end{aligned} \quad (12)$$

Similarly, if there exists $t_0 \in [\frac{L}{2}, L) \subset [2\varepsilon, L)$ with $a_2(t_0) \geq \frac{1}{2}a_2(0)$, we would have (with $a_2(t_0) - a_2(0) \geq -\frac{1}{2}a_2(0) > 0$)

$$R_\varepsilon(f) \geq \int_0^{t_0} a_2'(t)^2 h_\varepsilon^{n-1}(t) dt \geq \frac{1}{16\varepsilon} a_2(0)^2 \geq \frac{1}{16\varepsilon} \frac{1}{2b(b-1)^2} \frac{|\Sigma_1|}{|\Sigma_2|}. \quad (13)$$

Let us assume now that for each $t \in [\frac{L}{2}, L)$, $a_1(t) \geq \frac{1}{2}a_1(0)$ and $a_2(t) \leq \frac{1}{2}a_2(0)$. We then have, taking into account that, for each $j \leq b$, $\int_{\Sigma_j} \phi_i dv_{g_0} = 0$ if $i \neq j$ and $\int_{\Sigma_j} \phi_j dv_{g_0} = |\Sigma_j|^{\frac{1}{2}}$,

$$\int_{\Sigma_1 \times [\frac{L}{2}, L)} f(x, t) dv_g = \frac{|\Sigma_1|^{\frac{1}{2}}}{|\Sigma_1 \times [\frac{L}{2}, L)]_g} \int_{\frac{L}{2}}^L a_1(t) dt \geq \frac{1}{2|\Sigma_1|^{\frac{1}{2}}} a_1(0) > 0$$

and

$$\int_{\Sigma_2 \times [\frac{L}{2}, L)} f(x, t) dv_g = \frac{|\Sigma_2|^{\frac{1}{2}}}{|\Sigma_2 \times [\frac{L}{2}, L)]_g} \int_{\frac{L}{2}}^L a_2(t) dt \leq \frac{1}{2|\Sigma_2|^{\frac{1}{2}}} a_2(0) < 0.$$

We apply Lemma 3.4 to the function f in the complement Ω in M of $\Sigma \times [0, \frac{L}{2})$. This leads to

$$\int_{\Omega} |df|^2 dv_g \geq \frac{L\mu(\Omega, g)}{4} \min_{j \leq b} |\Sigma_j| \left(\frac{1}{2|\Sigma_1|^{\frac{1}{2}}} a_1(0) - \frac{1}{2|\Sigma_2|^{\frac{1}{2}}} a_2(0) \right)^2$$

where $\mu(\Omega, g)$ is the first positive Neumann eigenvalue of the Laplacian in Ω . One has

$$\begin{aligned} \frac{1}{|\Sigma_1|^{\frac{1}{2}}} a_1(0) - \frac{1}{|\Sigma_2|^{\frac{1}{2}}} a_2(0) &\geq \frac{1}{|\Sigma_1|^{\frac{1}{2}}} \frac{1}{\sqrt{2b}} + \frac{1}{|\Sigma_2|} \frac{|\Sigma_1|^{\frac{1}{2}}}{(b-1)\sqrt{2b}} \\ &= \frac{|\Sigma_1|^{\frac{1}{2}}}{\sqrt{2b}} \left(\frac{1}{|\Sigma_1|} + \frac{1}{(b-1)|\Sigma_2|} \right) \geq \frac{b}{(b-1)\sqrt{2b}} \frac{\min_{j \leq b} |\Sigma_j|^{\frac{1}{2}}}{\max_{j \leq b} |\Sigma_j|}. \end{aligned}$$

Thus,

$$\int_{\Omega} |df|^2 dv_g \geq \frac{bL}{32(b-1)^2} \mu(\Omega, g) \frac{\min_{j \leq b} |\Sigma_j|^2}{\max_{j \leq b} |\Sigma_j|^2}.$$

Since $h_{\varepsilon} = \varepsilon^{-2}$ on Ω , we get

$$\begin{aligned} R(f_{\varepsilon}) &= \int_M |df|^2 h_{\varepsilon}^{n-1} dv_g \\ &\geq \varepsilon^{-2(n-1)} \int_{\Omega} |df|^2 dv_g \geq \frac{\mu(\Omega, g)bL}{32(b-1)^2} \frac{\min_{j \leq b} |\Sigma_j|^2}{\max_{j \leq b} |\Sigma_j|^2} \times \frac{1}{\varepsilon} \end{aligned} \quad (14)$$

Combining (12), (13) and (14) leads to (11).

In conclusion, the inequality of the proposition holds with $C = \min\{A, B\}$. \square

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let (M, g) be a compact Riemannian manifold with $b > 0$ boundary components. Given $\delta > 0$, define

$$N(\delta) = \{p \in M : d_g(p, \Sigma) < \delta\}.$$

The normal exponential map along the boundary defines Fermi coordinates on a neighbourhood V of the boundary. The distance t to the boundary is one of the coordinates. It follows from the Gauss Lemma that the Riemannian metric is expressed by $g = h_t + dt^2$, where h_t is the restriction of g to the parallel hypersurface Σ_t at distance t from the boundary Σ . In particular it follows from $h_0 = g_0$ and continuity that there exists $\delta > 0$ such that the restriction of h_t to Σ_t is quasi-isometric to g_0 with ratio 2 for each $t \in [0, 3\delta]$. Let $\chi : [0, 3\delta] \rightarrow \mathbb{R}$ be a smooth

non-decreasing function with value 0 on $[0, \delta]$ and value 1 on $[2\delta, 3\delta]$. The metric g_δ is defined, using Fermi coordinates, by

$$g_\delta = \begin{cases} \chi(t)(h_t + dt^2) + (1 - \chi(t))(g_0 + dt^2) & \text{in } N(3\delta), \\ g & \text{elsewhere.} \end{cases}$$

The metric g_δ is quasi-isometric to g with ratio 2, and it is isometric to the product metric $g_0 + dt^2$ on $N(\delta)$. One can now apply Proposition 3.1 and Proposition 3.3 to (M, g_δ) . In both cases, this leads to a family of smooth functions h_ϵ such that some eigenvalue σ_j satisfies $\lim_{\epsilon \rightarrow 0} \sigma_j(M, h_\epsilon^2 g_\delta) = +\infty$. Note that $h_\epsilon^2 g$ is quasi-isometric to $h_\epsilon^2 g_\delta$ with ratio 2. Therefore, one can apply Proposition 2.2 to deduce that $\lim_{\epsilon \rightarrow 0} \sigma_j(M, h_\epsilon^2 g) = +\infty$. \square

The conformal perturbation $g_\epsilon = h_\epsilon^2 g$ that is used in the proof of Proposition 3.1 is so that the diameter of (M, g_ϵ) tends to infinity as $\epsilon \rightarrow 0$. Moreover, when $b \geq 2$, then the distance between the components of the boundary also goes to infinity as $\epsilon \rightarrow 0$. In the following theorem, a family of metrics on a cylinder is constructed, which coincide with the initial one along the boundary and such that the distance between the two components of the boundary is independent of ϵ , while σ_2 becomes arbitrarily large.

Theorem 3.5. *Let (Σ, g_0) be a closed connected Riemannian manifold of dimension $n \geq 2$ and consider the cylinder $M = \Sigma \times [-L, L]$, with $L > 0$, endowed with the product metric $g = g_0 + dt^2$. For every $\epsilon > 0$, sufficiently small, there exists a Riemannian metric $g_\epsilon = h_\epsilon^2(t)g_0 + dt^2$ which coincides with g in a neighborhood of Σ in M and such that*

$$\sigma_2(g_\epsilon) \geq \frac{C}{\epsilon},$$

where $C = \frac{1}{4} \min(\lambda_2(\Sigma), \frac{1}{6})$. The distance between the two boundary components is independent of ϵ and may be chosen arbitrarily small.

Proof. Given $0 < \epsilon < \frac{1}{4} \min(L, \frac{1}{L})$, let $h_\epsilon : [-L, L] \rightarrow [1, +\infty)$ be a smooth even function such that

$$h_\epsilon(t) = \begin{cases} 1 & \text{in } [-L, -L + \epsilon] \cup [L - \epsilon, L], \\ \epsilon^{-2} & \text{in } [-L + 2\epsilon, L - 2\epsilon]. \end{cases}$$

On the manifold $M = \Sigma \times [-L, L]$, the metric g_ϵ defined by

$$g_\epsilon(t, p) = h_\epsilon^2(t)g_0 + dt^2.$$

Let f be a smooth function on $M = \Sigma \times [-L, L]$ with $\int_{\partial M} f dv_{g_0} = 0$ and such that $\int_{\partial M} f^2 dv_{g_0} = 1$. As before, we develop the function f in

Fourier series

$$f(x, t) = \sum_{j \geq 1} a_j(t) \phi_j(x) \quad (15)$$

where $\{\phi_j\}_{j \in \mathbb{N}}$ is an orthonormal basis of eigenfunctions of the Laplacian on Σ , with $\Delta \phi_j = \lambda_j \phi_j$. The eigenfunction corresponding to $\lambda_1 = 0$ is the constant function ϕ_1 on Σ and $\int_{\Sigma} \phi_j dv_{g_0} = 0$ for all $j \geq 2$. The conditions $\int_{\partial M} f dv_{g_0} = \int_{\Sigma} (f(x, L) + f(x, -L)) dv_{g_0} = 0$ and $\int_{\partial M} f^2 dv_{g_0} = \int_{\Sigma} (f(x, L)^2 + f(x, -L)^2) dv_{g_0} = 1$ amount to

$$a_1(-L) + a_1(L) = 0 \quad , \quad \sum_{j=1}^{\infty} (a_j^2(-L) + a_j^2(L)) = 1.$$

From $df(x, t) = \sum_{j=1}^{\infty} (a'_j(t) \phi_j(x) dt + a_j(t) d_0 \phi_j(x))$ we get

$$R_{\varepsilon}(f) = \int_M |df|_{g_{\varepsilon}}^2 dv_{g_{\varepsilon}}$$

where

$$|df|_{g_{\varepsilon}}^2 = \sum_j a'_j(t)^2 \phi_j(x)^2 + h_{\varepsilon}^{-2}(t) a_j(t)^2 |d_0 \phi_j(x)|_{g_0^2},$$

and $dv_{g_{\varepsilon}} = h_{\varepsilon}^n dv_g$ so that

$$R_{\varepsilon}(f) = \sum_{j=1}^{\infty} \int_{-L}^L (a_j'^2(t) h_{\varepsilon}^n(t) + \lambda_j a_j^2(t) h_{\varepsilon}^{n-2}(t)) dt.$$

We set $R_{\varepsilon}^j(f) = \int_{-L}^L (a_j'^2(t) h_{\varepsilon}^n(t) + \lambda_j a_j^2(t) h_{\varepsilon}^{n-2}(t)) dt$ so that $R_{\varepsilon}(f) = \sum_{j=1}^{\infty} R_{\varepsilon}^j(f)$.

Step 1 : For all $j \geq 2$, we will prove that

$$R_{\varepsilon}^j(f) \geq \frac{A}{\varepsilon} (a_j^2(-L) + a_j^2(L))$$

with $A = \min(\frac{\lambda_2}{4}, \frac{1}{12})$.

Indeed, let us fix an integer j . If there exists $t_0 \in [-L + 2\varepsilon, -L + 3\varepsilon]$ with $|a_j(t_0) - a_j(-L)| \geq \frac{1}{2}|a_j(-L)|$, then, using the Cauchy-Schwarz inequality

$$\begin{aligned} \int_{-L}^0 a_j'^2(t) h_{\varepsilon}^n(t) dt &\geq \int_{-L}^{t_0} a_j'^2(t) dt \geq \frac{1}{t_0 + L} \left(\int_{-L}^{t_0} a'_j(t) dt \right)^2 \\ &\geq \frac{1}{3\varepsilon} (a_j(t_0) - a_j(-L))^2 \geq \frac{1}{12\varepsilon} a_j^2(-L). \end{aligned} \quad (16)$$

Otherwise, $|a_j(t) - a_j(-L)| \leq \frac{1}{2}|a_j(-L)|$ for all $t \in [-L + 2\varepsilon, -L + 3\varepsilon]$, which implies $|a_j(t)| \geq \frac{1}{2}|a_j(-L)|$ and then

$$\int_{-L}^0 \lambda_j a_j^2(t) h_\varepsilon^{n-2}(t) dt \geq \int_{-L+2\varepsilon}^{-L+3\varepsilon} \lambda_j a_j^2(t) \varepsilon^{-2(n-1)} dt \geq \frac{\lambda_j}{4\varepsilon^{2n-3}} a_j^2(-L). \quad (17)$$

Thus, in all cases, we have for $j \geq 2$ (with $\varepsilon^{2n-3} \leq \varepsilon$ and $\lambda_j \geq \lambda_2$)

$$\int_{-L}^0 (a_j'^2(t) h_\varepsilon^n(t) + \lambda_j a_j^2(t) h_\varepsilon^{n-2}(t)) dt \geq \frac{A}{\varepsilon} a_j^2(-L) \quad (18)$$

with $A = \min(\frac{\lambda_2}{4}, \frac{1}{12})$. The same arguments lead to

$$\int_0^L (a_j'^2(t) h_\varepsilon^n(t) + \lambda_j a_j^2(t) h_\varepsilon^{n-2}(t)) dt \geq \frac{A}{\varepsilon} a_j^2(L). \quad (19)$$

Combining (18) and (19), we get for all $j \geq 2$,

$$R_\varepsilon^j(f) \geq \frac{A}{\varepsilon} (a_j^2(-L) + a_j^2(L)). \quad (20)$$

Step 2 For $j = 1$, we will also show that

$$R_\varepsilon^1(f) \geq \frac{1}{24\varepsilon} (a_1^2(-L) + a_1^2(L)).$$

Indeed, recall that we have $a_1(-L) = -a_1(L)$. Using (16), we see that if there exists $t_0 \in [-L + 2\varepsilon, -L + 3\varepsilon]$ with $|a_1(t_0) - a_1(-L)| \geq \frac{1}{2}|a_1(-L)|$, then

$$R_\varepsilon^1(f) \geq \int_{-L}^0 a_1'^2(t) h_\varepsilon^n(t) dt \geq \frac{1}{12\varepsilon} a_1^2(-L) = \frac{1}{24\varepsilon} (a_1^2(-L) + a_1^2(L)).$$

Similarly, if there exists $t_1 \in [L - 3\varepsilon, L - 2\varepsilon]$ with $|a_1(t_1) - a_1(L)| \geq \frac{1}{2}|a_1(L)|$, then

$$R_\varepsilon^1(f) \geq \int_0^L a_1'^2(t) h_\varepsilon^n(t) dt \geq \frac{1}{12\varepsilon} a_1^2(L) = \frac{1}{24\varepsilon} (a_1^2(-L) + a_1^2(L)).$$

Now, assume that we have both $|a_1(t) - a_1(-L)| \leq \frac{1}{2}|a_1(-L)|$ for all $t \in [-L + 2\varepsilon, -L + 3\varepsilon]$ and $|a_1(L) - a_1(t)| \leq \frac{1}{2}|a_1(L)|$ for all $t \in [L - 3\varepsilon, L - 2\varepsilon]$. Using the Cauchy-Schwarz inequality we get

$$\begin{aligned} R_\varepsilon^1(f) &\geq \int_{-L+3\varepsilon}^{L-3\varepsilon} a_1'^2(t) h_\varepsilon^n(t) dt = \varepsilon^{-2n} \int_{-L+3\varepsilon}^{L-3\varepsilon} a_1'^2(t) dt \geq \frac{\varepsilon^{-2n}}{2L - 6\varepsilon} \left(\int_{-L+3\varepsilon}^{L-3\varepsilon} a_1'(t) dt \right)^2 \\ &\geq \frac{\varepsilon^{-2n}}{2L} (a_1(L - 3\varepsilon) - a_1(-L + 3\varepsilon))^2 \end{aligned}$$

Assume w.l.o.g. $a_1(L) > 0$. Then, under the conditions above, we have

$$a_1(L - 3\varepsilon) \geq a_1(L) - |a_1(L) - a_1(L - 3\varepsilon)| \geq \frac{1}{2}a_1(L)$$

and

$$a_1(-L + 3\varepsilon) \leq a_1(-L) + |a_1(-L + 3\varepsilon) - a_1(-L)| \leq a_1(-L) + \frac{1}{2}|a_1(-L)| = -\frac{1}{2}a_1(L).$$

Therefore, $a_1(L - 3\varepsilon) - a_1(-L + 3\varepsilon) \geq a_1(L)$ and, then

$$R_\varepsilon^1(f) \geq \frac{\varepsilon^{-2n}}{2L} a_1^2(L) = \frac{\varepsilon^{-2n}}{4L} (a_1^2(-L) + a_1^2(L)) \geq \frac{1}{\varepsilon} (a_1^2(-L) + a_1^2(L))$$

In conclusion, we have

$$R_\varepsilon(f) = \sum_{j \geq 1} R_\varepsilon^j(f) \geq \frac{C}{\varepsilon} \sum_{j \geq 1} (a_j^2(-L) + a_j^2(L)) = \frac{C}{\varepsilon}$$

with $C = \min(A, \frac{1}{24}) = \min(\frac{\lambda_2}{4}, \frac{1}{24})$. □

Remark 3.6. *This example shows that if we are far from being a product", then, immediately, large eigenvalues could appear. Also, in our construction, the natural projection from $(M, g_\varepsilon) \rightarrow [-L, L]$ is a Riemannian submersion on $[-L, L]$. This implies that $2L$ is also the distance between the two boundaries in (M, g_ε) , for any ε . It is interesting to see that this distance could be very small, without small eigenvalues.*

REFERENCES

- [1] M. S. Agranovich. On a mixed Poincaré-Steklov type spectral problem in a Lipschitz domain. *Russ. J. Math. Phys.*, 13(3):239–244, 2006.
- [2] R. Bañuelos, T. Kulczycki, I. Polterovich, and B. Siudeja. Eigenvalue inequalities for mixed Steklov problems. In *Operator theory and its applications*, volume 231 of *Amer. Math. Soc. Transl. Ser. 2*, pages 19–34. Amer. Math. Soc., Providence, RI, 2010.
- [3] B. Colbois, A. El Soufi, and A. Girouard. Isoperimetric control of the Steklov spectrum. *J. Funct. Anal.*, 261(5):1384–1399, 2011.
- [4] B. Colbois, A. Girouard, and B. Raveendran. The Steklov spectrum and coarse discretizations of manifolds with boundary. *To appear*, 2017.
- [5] J. Dodziuk. Eigenvalues of the Laplacian on forms. *Proc. Amer. Math. Soc.*, 85(3):437–443, 1982.
- [6] J. F. Escobar. The geometry of the first non-zero Stekloff eigenvalue. *J. Funct. Anal.*, 150(2):544–556, 1997.
- [7] A. Girouard, L. Parnowski, I. Polterovich, and D. A. Sher. The Steklov spectrum of surfaces: asymptotics and invariants. *Math. Proc. Cambridge Philos. Soc.*, 157(3):379–389, 2014.
- [8] A. Girouard and I. Polterovich. Spectral geometry of the Steklov problem. *To appear in Journal of Spectral Theory*. arXiv:1411.6567.

- [9] A. Hassannezhad. Conformal upper bounds for the eigenvalues of the Laplacian and Steklov problem. *J. Funct. Anal.*, 261(12):3419–3436, 2011.
- [10] P. Jammes. Une inégalité de Cheeger pour le spectre de Steklov. *Ann. Inst. Fourier (Grenoble)*, 65(3):1381–1385, 2015.
- [11] G. Kokarev. Variational aspects of Laplace eigenvalues on Riemannian surfaces. *Adv. Math.*, 258:191–239, 2014.
- [12] L. Provenzano and J. Stubbe. Weyl-type bounds for Steklov eigenvalues. *ArXiv e-prints*, November 2016.
- [13] Q. Wang and C. Xia. Isoperimetric bounds for the first eigenvalue of the laplacian. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, 2009.

UNIVERSITÉ DE NEUCHÂTEL, INSTITUT DE MATHÉMATIQUES, RUE EMILE-ARGAND 11, CH-2000 NEUCHÂTEL, SWITZERLAND
E-mail address: `bruno.colbois@unine.ch`

LABORATOIRE DE MATHÉMATIQUES ET PHYSIQUE THÉORIQUE, UMR-CNRS 6083, UNIVERSITÉ FRANÇOIS RABELAIS DE TOURS, PARC DE GRANDMONT, 37200 TOURS, FRANCE
E-mail address: `elsoufi@univ-tours.fr`

DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE, PAVILLON ALEXANDRE-VACHON, UNIVERSITÉ LAVAL, QUÉBEC, QC, G1V 0A6, CANADA
E-mail address: `Alexandre.Girouard@ulaval.ca`